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Transformations of the equations of motion for the unsteady rectilinear flow of a perfect gas

J. A. STEKETEE

Department of Aeronautical Engineering, Delft University of Technology, Delft, The Netherlands

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SUMMARY

Unsteady rectilinear flows of a perfect gas have usually been studied in the Eulerian x, t-plane and less frequently in Lagrangian or other independent variables. In the homentropic case, when Riemann invariants exist, these may be used with advantage as independent variables.

In the present paper some simple techniques are applied to the equations of motion in Lagrangian variables in order to obtain a survey of the equations of motion when the independent variables are varied in a systematic fashion. This transformation theory reveals some connections in the problem, which appeared in the literature in a fragmentary and puzzling fashion. Also several results obtained before are simplified, unified and hence clarified by the present treatment. No new and detailed solutions are presented but it is expected that the study will be of some help for further work on unsteady rectilinear flows.

1. Introduction

It is the purpose of the following lines to develop in a systematic manner a number of different forms for the equations describing the unsteady rectilinear motion of a perfect gas.

The equations of motion for these flows, have been studied extensively, together with applications, in several books [1, 2, 3, 4] and in numerous papers (see for example [5, 6, 7]). It is usual in these discussions to start from the equations of motion in the x, t-plane and not so common to select Lagrangian or other coordinates as independent variables. In the homentropic case the characteristic equations admit Riemann-invariants and these can be used with advantage as independent variables. In the non-homentropic case however this method fails and the characteristics are, at least analytically, less effective.

The problem of obtaining further information on non-homentropic flows formed the starting point for the present investigation. In the discussion to be presented the characteristics in first instance, play no part.

Starting from the equations of motion in Lagrangian form the independent variables are systematically changed. Also some simple properties suggested by the Lagrangian equations are systematically exploited. By allowing "nature to take its course" a number of alternative equations then appear without effort. Several of these equations can be selected as basis for the formulation of the problem, provided they would be supplemented in a suitable fashion with initial- and boundary-conditions. Some of these equations have appeared in the literature.

The transformation theory so obtained reveals some connections in the problem, which appeared in the literature in a fragmentary and puzzling fashion. Also several results, obtained before after a good deal of not very transparent manipulation, are obtained now with little effort. No new and detailed solutions are presented in this paper but it contributes to the simplification of the theory and, the clarification of its structure. It is expected to be of some help for further work in this field.

The paper is divided in eight sections. In section 2 the equations of motion are presented in the Lagrangian form. The two independent variables are h, labelling the fluid elements and the time t. In section 3 the independent variables h, t are replaced by new variables ξ, η and the equations of motion take the form of Jacobians. Selecting for ξ, η any pair of the original flow parameters u, p, V, h and t ten sets of equations of motion appear, which are listed in Table 1 and for the homentropic case in Table 2.

In the latter case only six distinct sets of equations appear since p and V are functionally related.

The Lagrangian equations of motion are of a form which invites the introduction of "potentials" or "stream functions". In section 4 two functions E and K are introduced, which identically satisfy the equations of mass respectively momentum-conservation. Between the first derivatives of E and K there exists one relation, which again can be satisfied identically by a function $\Phi(h, t)$ introduced in section 5. The problem of the unsteady rectilinear flow of a gas is now reduced to the determination of one function Φ. The flow parameters u, p and V are expressed as second derivatives of Φ.

Other possibilities to reduce the problem to the solution of one equation appear in section 6 where the Legendre transformations of E and K are introduced. In first instance this leads to 8 different equations. Further possibilities appear if in these equations the single dependent variable is interchanged with the independent variables.

In section 7 also the Legendre Transformations of $\Phi(h, t)$ are introduced, which again leads to some new equations. Some relations between the Legendre transformations in section 7 and those in section 6 are established. Finally in section 8 some attention is paid to the characteristics of the 10 sets of equations in Table 1. The characteristics may be written in identical form for the ten sets. The Riemann invariants of the homentropic case are considered, while one of the Legendre transformations of $K(h, t)$ enables us to construct in a simple fashion the generalized Riemann invariants for a Martin–Ludford gas.

In the homentropic case it is usual to select the Riemann-invariants as independent variables. The problem can then be reduced to a linear equation of Euler–Poisson–Darboux type. It is shown that this equation is closely related to some equations in section 6 for the Legendre transformations of E and K, provided the latter are specialized to the homentropic case.

2. The Lagrangian equations of motion

The flows to be studied are unsteady and rectilinear as in the shock-tube or in a gun. Two independent variables are needed to describe the motion and since we intend to follow the Lagrangian description, it is suitable to select a coordinate s labelling the fluid elements and a second coordinate t to denote the time. The position of a fluid element s at time t will be given by the Cartesian coordinate x, along an axis in the direction of the motion. We write:

$$x = x(s, t). \tag{2.1}$$

The choice of the parameter s can be made in a variety of ways. Taking for s the position of the fluid elements at time $t = 0$, we have:

$$s \overset{\text{def}}{=} x(s, 0). \tag{2.2}$$

The equations of motion are the three conservation laws of mass, momentum and energy. They may be written in the form:

$$\frac{\partial}{\partial t}(\rho\, dx) = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho u\, dx) = -\frac{\partial p}{\partial x}\, dx, \tag{2.4}$$

$$\frac{\partial}{\partial t}\{\rho\, dx(U + \tfrac{1}{2}u^2)\} = -\frac{\partial}{\partial x}(up)dx, \tag{2.5}$$

with ρ denoting the density, p the pressure, U the internal energy per unit mass and $u = \partial x/\partial t$, the speed of a fluid element. Effects of viscosity and heat conduction are assumed to be negligible throughout. It may be noticed that x in the partial derivatives on the R.H.S. in (2.4) and (2.5) is not a Lagrangian variable. Denoting the density at time $t = 0$ by $\rho_0(s)$ one finds from (2.3):

$$\rho\, dx = \rho_0\, ds, \qquad \rho_0(s) = \rho(s, t)\frac{\partial x}{\partial s}. \tag{2.6}$$

By means of (2.6) the equations (2.4) and (2.5) can be rewritten in the Lagrangian variables in the form:

$$\rho_0(s)\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial s}, \tag{2.7}$$

$$\rho_0(s)\frac{\partial}{\partial t}(U + \tfrac{1}{2}u^2) = -\frac{\partial}{\partial s}(up). \tag{2.8}$$

Assuming the gas to be an ideal gas with constant specific heats, employing (2.7), the derivative to t of (2.6) and the second law of thermodynamics in the form:

$$T\frac{\partial S}{\partial t} = \frac{\partial U}{\partial t} + p\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right), \tag{2.9}$$

(S is entropy per unit mass) one can reduce (2.8) to the pair of relations:

$$\frac{\partial S}{\partial t} = 0, \qquad p = \rho^\gamma \exp\left(\frac{S - S_0}{c_v}\right), \tag{2.10}$$

with c_v representing the specific heat at constant volume and γ the ratio of the specific heats at constant pressure and constant volume, both assumed constant. Finally S_0 is a constant of integration, which may be incorporated in S.

It is convenient to replace the coordinate s by a coordinate h, defined by the mass increment:

$$dh = \rho_0(s)ds. \tag{2.11}$$

Since the entropy S depends on s only, as indicated by (2.10), it is convenient to write:

$$B(h) = \exp\left(\frac{S - S_0}{c_v}\right). \tag{2.12}$$

It is of crucial importance to distinguish between homentropic and non-homentropic flows. In each type of flow a fluid element will retain its entropy for all time. In the homentropic case each unit mass has the same entropy and $B(h)$ reduces to a constant. In the non-homentropic case the value of the entropy varies from one unit mass to another. Mathematically speaking $B(h)$ can à priori be considered as an arbitrary function of h. In order to construct solutions of the equations however some forms of $B(h)$ will be easier to handle than others, while physically speaking it would seem that the forms of $B(h)$ occurring in reality are not, as a rule, arbitrary.

Introducing the specific volume $V = \rho^{-1}$, the equations (2.6), (2.7) and (2.10) can now be written:

$$\begin{aligned} &V = \frac{\partial x}{\partial h}, \qquad \frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial h}, \\ &\frac{\partial B}{\partial t} = 0, \qquad pV^\gamma = B(h). \end{aligned} \tag{2.13}$$

A more symmetric form, which is preferred, is obtained by employing u instead of x in the first two equations of (2.13). Differentiation to t of the first equation in (2.13) and substitution of u then leads to:

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0, \tag{2.14}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0, \tag{2.15}$$

$$\frac{\partial B}{\partial t} = 0, \qquad pV^\gamma = B(h). \tag{2.16}$$

When $B(h)$ has been chosen (2.14)–(2.16) forms a set of three equations for the determination of u, p and V. It is easily seen that the equations are non-linear.

To complete this section some familiar relations are derived, which will be needed occasionally. The speed of sound a is defined by:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S=\mathrm{const}} = -V^2\left(\frac{\partial p}{\partial V}\right)_{S=\mathrm{const}}. \tag{2.17}$$

In order to keep S constant, when differentiating in (2.17), h has to be kept constant in the non-homentropic case. One finds:

$$
\begin{aligned}
a^2 &= \gamma B(h) V^{-(\gamma-1)} = \gamma p V, \\
\frac{a^2}{V^2} &= \gamma B(h) V^{-(\gamma+1)} = \gamma \frac{p}{V}.
\end{aligned}
\tag{2.18}
$$

The expression $a/V = \rho a$ is the local characteristic acoustic impedance. In the homentropic case $B(h) = \text{const.}$ and a depends on V (or p) alone. In the non-homentropic case the speed of sound depends explicitly on the variable h also.

Eliminating V from (2.14) by means of (2.16), the equations (2.14) and (2.15) can be written:

$$
\frac{V^2}{a^2} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial h} = 0, \qquad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0. \tag{2.19}
$$

The characteristic equations can now easily be constructed. One finds the two sets of characteristic equations:

$$
\left.
\begin{aligned}
dh - \frac{a}{V} dt &= 0, \\
dp + \frac{a}{V} du &= 0,
\end{aligned}
\right\} \tag{2.20}
$$

$$
\left.
\begin{aligned}
dh + \frac{a}{V} dt &= 0, \\
dp - \frac{a}{V} du &= 0,
\end{aligned}
\right\} \tag{2.21}
$$

which represent sound waves, travelling with the speed of sound with respect to the non-uniform, moving gas.

3. The equations in the form of Jacobians. The ten sets

Instead of the Lagrangian variables (h, t) new independent variables (ξ, η) will be selected. Substituting these new variables one finds in the familiar way:

$$
\begin{aligned}
\frac{\partial}{\partial h} &= \frac{1}{J}\left(\frac{\partial}{\partial \xi} \frac{\partial t}{\partial \eta} - \frac{\partial t}{\partial \xi} \frac{\partial}{\partial \eta}\right), \\
\frac{\partial}{\partial t} &= -\frac{1}{J}\left(\frac{\partial}{\partial \xi} \frac{\partial h}{\partial \eta} - \frac{\partial h}{\partial \xi} \frac{\partial}{\partial \eta}\right),
\end{aligned}
\tag{3.1}
$$

with

$$
J = \frac{\partial(h, t)}{\partial(\xi, \eta)} = \frac{\partial h}{\partial \xi} \frac{\partial t}{\partial \eta} - \frac{\partial t}{\partial \xi} \frac{\partial h}{\partial \eta}. \tag{3.2}
$$

It will be assumed that $J \neq 0$ and bounded. Application of (3.1) to (2.14)–(2.16) then leads to the equations of motion in the form of Jacobians. One obtains:

$$\frac{\partial(V,h)}{\partial(\xi,\eta)} + \frac{\partial(u,t)}{\partial(\xi,\eta)} = 0,$$
$$\frac{\partial(u,h)}{\partial(\xi,\eta)} - \frac{\partial(p,t)}{\partial(\xi,\eta)} = 0, \tag{3.3}$$
$$\frac{\partial(B,h)}{\partial(\xi,\eta)} = 0.$$

and upon multiplication with $d\xi d\eta$:

$$dVdh + dudt = 0,$$
$$dudh - dpdt = 0, \tag{3.4}$$
$$dBdh = 0.$$

It should be noted that the differentials in (3.4) do not commute. One has:

$$dtdu = -dudt, \tag{3.5}$$

owing to the properties of the Jacobians.

In the equations of motion, so far derived, the 5 parameters u, p, V, h and t appeared. For the new independent variables (ξ, η) one may select any pair of these 5 parameters. Ten pairs of independent variables can be selected in this way, leading to ten sets of equations of motion. These equations can be easily written out and are listed in Table 1. It should be noted, that the ten sets apply in the non-homentropic case, when $B(h) \neq$ constant.

In those cases in the Table, where the pair of independent variables is taken from the triplet p, V, h, the third parameter of the triplet should not be considered as a dependent variable. For example Set II is a set of equations for the determination of u and t as functions of (p, h). Similar arguments apply to the Sets IV and IX.

In those cases, where one independent variable is selected from the triplet p, V, h, there is a choice, which of the two remaining elements in the triplet is to be considered as dependent variable. In Set I for example one may take u and p as dependent variables, but equally well, if desired, u and V. Similar arguments apply to the Sets III, V, VII, VIII and X.

In Set VI finally one may select any pair of the triplet p, V, h as dependent variables.

The different forms, appearing when different selections of the dependent variables are made, are easily obtained when making the required substitutions, by means of (2.16) in the equations of Table 1.

In the homentropic case $B(h) =$ const. and it is clear from (2.16) that p and V cannot be taken as independent variables. In this case the pairs of independent variables have to be selected from either the quadruplet u, p, h, t or u, V, h, t. Clearly from each quadruplet only 6 pairs can be selected, leading to 6 different sets of equations of motion. Comparing these results with Table 1 it is clear, that Set IX vanishes, while the Sets II and IV, V and VII, VIII and X are identical. This may be easily verified since we have in the homentropic case:

$$dp = -\frac{a^2}{V^2}dV, \tag{3.6}$$

with a the speed of sound.

TABLE 1

Non homentropic equations

I h, t	$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0$		E
	$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0$	$\frac{\partial B}{\partial t} = 0$	K
II h, p	$-\frac{\partial V}{\partial p} + \frac{\partial u}{\partial h}\frac{\partial t}{\partial p} - \frac{\partial t}{\partial h}\frac{\partial u}{\partial p} = 0$		
	$\frac{\partial u}{\partial p} - \frac{\partial t}{\partial h} = 0$	$\frac{\partial B}{\partial p} = 0$	M
III h, u	$\frac{\partial V}{\partial u} + \frac{\partial t}{\partial h} = 0$		F
	$1 + \frac{\partial p}{\partial h}\frac{\partial t}{\partial u} - \frac{\partial t}{\partial h}\frac{\partial p}{\partial u} = 0$	$\frac{\partial B}{\partial u} = 0$	
IV h, V	$-1 + \frac{\partial u}{\partial h}\frac{\partial t}{\partial V} - \frac{\partial t}{\partial h}\frac{\partial u}{\partial V} = 0$		
	$\frac{\partial u}{\partial V} + \frac{\partial p}{\partial h}\frac{\partial t}{\partial V} - \frac{\partial t}{\partial h}\frac{\partial p}{\partial V} = 0$	$\frac{\partial B}{\partial V} = 0$	
V t, p	$\frac{\partial V}{\partial t}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial t}\frac{\partial V}{\partial p} - \frac{\partial u}{\partial p} = 0$		
	$\frac{\partial u}{\partial t}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial t}\frac{\partial u}{\partial p} + 1 = 0$	$B = B(h)$	
VI t, u	$\frac{\partial V}{\partial t}\frac{\partial h}{\partial u} - \frac{\partial h}{\partial t}\frac{\partial V}{\partial u} - 1 = 0$		
	$\frac{\partial h}{\partial t} - \frac{\partial p}{\partial u} = 0$	$B = B(h)$	L
VII t, V	$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial V} = 0$		G
	$\frac{\partial u}{\partial t}\frac{\partial h}{\partial V} - \frac{\partial h}{\partial t}\frac{\partial u}{\partial V} + \frac{\partial p}{\partial V} = 0$	$B = B(h)$	
VIII p, u	$\frac{\partial V}{\partial p}\frac{\partial h}{\partial u} - \frac{\partial h}{\partial p}\frac{\partial V}{\partial u} - \frac{\partial t}{\partial p} = 0$		
	$\frac{\partial t}{\partial u} + \frac{\partial h}{\partial p} = 0$	$B = B(h)$	N
IX p, V	$-\frac{\partial h}{\partial p} + \frac{\partial u}{\partial p}\frac{\partial t}{\partial V} - \frac{\partial t}{\partial p}\frac{\partial u}{\partial V} = 0$		
	$\frac{\partial u}{\partial p}\frac{\partial h}{\partial V} - \frac{\partial h}{\partial p}\frac{\partial u}{\partial V} - \frac{\partial t}{\partial V} = 0$	$B = B(h)$	

TABLE 1 (*continued*)

X V, u	$\frac{\partial h}{\partial u} - \frac{\partial t}{\partial V} = 0$ $-\frac{\partial h}{\partial V} - \frac{\partial p}{\partial V}\frac{\partial t}{\partial u} + \frac{\partial t}{\partial V}\frac{\partial p}{\partial u} = 0$	 $B = B(h)$	H

TABLE 2

Homentropic equations

I h, t	$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0$ $\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0$	E K
II h, p	$-\frac{dV}{dp} + \frac{\partial u}{\partial h}\frac{\partial t}{\partial p} - \frac{\partial t}{\partial h}\frac{\partial u}{\partial p} = 0$ $\frac{\partial u}{\partial p} - \frac{\partial t}{\partial h} = 0$	 M
III h, u	$\frac{\partial V}{\partial u} + \frac{\partial t}{\partial h} = 0$ $1 + \frac{\partial p}{\partial h}\frac{\partial t}{\partial u} - \frac{\partial t}{\partial h}\frac{\partial p}{\partial u} = 0$	F
IV h, V	$-1 + \frac{\partial u}{\partial h}\frac{\partial t}{\partial V} - \frac{\partial t}{\partial h}\frac{\partial u}{\partial V} = 0$ $\frac{\partial u}{\partial V} - \frac{dp}{dV}\frac{\partial t}{\partial h} = 0$	
V t, p	$\frac{dV}{dp}\frac{\partial h}{\partial t} + \frac{\partial u}{\partial p} = 0$ $\frac{\partial u}{\partial t}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial t}\frac{\partial u}{\partial p} + 1 = 0$	
VI t, u	$\frac{\partial V}{\partial t}\frac{\partial h}{\partial u} - \frac{\partial h}{\partial t}\frac{\partial V}{\partial u} - 1 = 0$ $\frac{\partial h}{\partial t} - \frac{\partial p}{\partial u} = 0$	 L
VII t, V	$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial V} = 0$ $\frac{\partial u}{\partial t}\frac{\partial h}{\partial V} - \frac{\partial h}{\partial t}\frac{\partial u}{\partial V} + \frac{dp}{dV} = 0$	G

TABLE 2 (*continued*)

VIII p, u	$\frac{dV}{dp}\frac{\partial h}{\partial u} - \frac{\partial t}{\partial p} = 0$ $\frac{\partial t}{\partial u} + \frac{\partial h}{\partial p} = 0$	N
IX p, V	-------	
X V, u	$\frac{\partial h}{\partial u} - \frac{\partial t}{\partial V} = 0$ $\frac{\partial h}{\partial V} + \frac{dp}{dV}\frac{\partial t}{\partial u} = 0$	H

The equations for the homentropic case have been listed in Table II. Owing to equations (2.16) p will depend only on V, or V on p and hence several equations in Table II are simpler than in Table 1. Also the identity of the Sets II and IV, etc. is easily checked upon inspection of Table 2.

In applications one must expect to find points, lines or regions where some of the Jacobians in this section do not satisfy the imposed conditions. The considerations therefore have a local character rather than a global significance.

4. The functions $E(h, t)$ and $K(h, t)$

Returning to the equations (2.14) and (2.15) it is noticed that their form suggests the introduction of a "potential" or "streamfunction". These functions may be introduced by remarking that an integration of the equations over a bounded domain in the h, t-plane can be reduced by means of Stokes' Theorem to an integral along the closed boundary of the domain of integration. Denoting the bounded domain by D and its boundary by C one has:

$$\iint_D \left(\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h}\right) dh\,dt = -\oint_c (V dh + u dt) = 0, \tag{4.1}$$

$$\iint_D \left(\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h}\right) dh\,dt = \oint_c (-u dh + p dt) = 0, \tag{4.2}$$

with the integral along the contour C taken in the anti-clockwise direction. An integration of the expressions $(Vdh + udt)$ and $(-udh + pdt)$ along a curve connecting two points in the h, t-plane will be, in a simply connected domain, independent of the shape of the curve, and depend only on the positions of the two endpoints. Hence the two expressions are exact differentials. This suggests the introduction of two functions $E(h, t)$ and $K(h, t)$

chosen in such a way that:

$$\begin{aligned} &dE = V dh + u dt, \quad dK = -u dh + p dt, \\ &E_h = V, \quad E_t = u = -K_h, \quad K_t = p, \end{aligned} \tag{4.3}$$

where partial derivatives have now been denoted by subscripts.

The function $E(h, t)$ is up to an arbitrary constant identical with the Cartesian coordinate $x(h, t)$ indicating the position of the fluid element, with label h at time t. This is easily verified from (2.13), the definition of velocity, and (4.3).

The function $K(h, t)$ is related to momentum. The function is introduced for the momentum equation, while udh and pdt represent small amounts of momentum. Another illuminating form can be obtained by considering h in the first equation of (4.3) as function of the Eulerian variables x ($= E$) and t, leading to:

$$dh = \frac{1}{V} dx - \frac{u}{V} dt = \rho dx - \rho u dt, \tag{4.4}$$

The function h in (4.4) identically satisfies the continuity equation in the Eulerian form. Substitution of (4.4) into the second equation of (4.3) then gives:

$$dK = -\rho u dx + (\rho u^2 + p) dt. \tag{4.5}$$

In this form $K(x, t)$ identically satisfies the Eulerian momentum equation in conservation form:

$$\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) = 0. \tag{4.6}$$

The problem of the unsteady rectilinear flow of a perfect gas can now be reformulated in terms of $E(h, t)$ and $K(h, t)$. Instead of the three equations (2.14)–(2.16) for p, V and u one has to solve the two equations for E and K:

$$\frac{\partial E}{\partial t} + \frac{\partial K}{\partial h} = 0, \tag{4.7}$$

$$\frac{\partial K}{\partial t} \left(\frac{\partial E}{\partial h} \right)^{\gamma} - B(h) = 0. \tag{4.8}$$

The first derivatives of E and K determine the flow parameters u, p and V according to (4.3). In order to represent a proper flow the functions E and K will in general be required to possess first and second derivatives, which are bounded and continuous. In the next section we shall proceed one step further and reduce the problem to the solution of one single equation for a function $\Phi(h, t)$.

In the remaining part of this section the Lagrangian coordinates (h, t) will be replaced by new independent variables (ξ, η) as in section 3. Employing the relation (3.1) and (3.2), assuming that $J \neq 0$ and bounded, the equations (4.7) and (4.8) assume the form:

$$\frac{\partial(h, E)}{\partial(\xi, \eta)} + \frac{\partial(K, t)}{\partial(\xi, \eta)} = 0, \tag{4.9}$$

$$\frac{\partial(h, K)}{\partial(\xi, \eta)}\left\{\frac{\partial(E, t)}{\partial(\xi, \eta)}\right\}^{\gamma} - B(h)\left\{\frac{\partial(h, t)}{\partial(\xi, \eta)}\right\}^{\gamma+1} = 0. \tag{4.10}$$

In the equations (4.7)–(4.10) the parameters E $(= x)$, K, h and t appear. Selecting any pair from this quadruplet as new independent variables (ξ, η), 6 different pairs can be selected leading to 6 different sets of equations. These equations have been listed in the first part of Table 3.

In order to display the connections between the partial derivatives in these equations and the parameters u, p and V which characterize the flow, the second part of Table 3 was constructed. The expressions with Set I represent essentially eq. (4.3), while the expressions for dh and dK in Set II are recognized as (4.4) and (4.5). The other expressions in the second part of Table 3 are obtained from similar simple manipulations with dE and dK in eq. (4.3).

5. The function $\Phi(h, t)$

Inspection of equation (4.7) shows that it has the same form as (2.14) and (2.15). The argument at the beginning of Section 4, leading to the introduction of E and K can therefore be repeated for equation (4.7) and we find:

$$\iint_D \left(\frac{\partial E}{\partial t} + \frac{\partial K}{\partial h}\right) dh\,dt = -\oint_c (E\,dh - K\,dt) = 0. \tag{5.1}$$

Next a function $\Phi(h, t)$ is introduced, in such a way that:

$$d\Phi = E\,dh - K\,dt, \qquad E = \Phi_h, \qquad K = -\Phi_t. \tag{5.2}$$

TABLE 3

1. *Equations of h, t, $E(=x)$, K*

I h, t	$\dfrac{\partial E}{\partial t} + \dfrac{\partial K}{\partial h} = 0$	$\dfrac{\partial K}{\partial t}\left(\dfrac{\partial E}{\partial h}\right)^{\gamma} - B(h) = 0$	Φ
II E, t	$\dfrac{\partial h}{\partial t} - \dfrac{\partial K}{\partial E} = 0$	$\dfrac{\partial(h, K)}{\partial(E, t)} - B(h)\left(\dfrac{\partial h}{\partial E}\right)^{\gamma+1} = 0$	ψ
III K, t	$\dfrac{\partial(h, E)}{\partial(K, t)} + 1 = 0$	$\dfrac{\partial h}{\partial t}\left(\dfrac{\partial E}{\partial K}\right)^{\gamma} + B(h)\left(\dfrac{\partial h}{\partial K}\right)^{\gamma+1} = 0$	
IV h, E	$1 + \dfrac{\partial(K, t)}{\partial(h, E)} = 0$	$\dfrac{\partial K}{\partial E}\left(-\dfrac{\partial t}{\partial h}\right)^{\gamma} - B(h)\left(\dfrac{\partial t}{\partial E}\right)^{\gamma+1} = 0$	
V h, K	$\dfrac{\partial E}{\partial K} - \dfrac{\partial t}{\partial h} = 0$	$\left\{\dfrac{\partial(E, t)}{\partial(h, K)}\right\}^{\gamma} - B(h)\left(\dfrac{\partial t}{\partial K}\right)^{\gamma+1} = 0$	θ
VI E, K	$\dfrac{\partial h}{\partial K} + \dfrac{\partial t}{\partial E} = 0$	$\dfrac{\partial h}{\partial E}\left(\dfrac{\partial t}{\partial K}\right)^{\gamma} - B(h)\left\{\dfrac{\partial(h, t)}{\partial(E, K)}\right\}^{\gamma+1} = 0$	χ

TABLE 3

2. *The physical parameters*

I h, t	$dE = Vdh + udt$ $dK = -udh + pdt$	$V = E_h$ $p = K_t$	$u = E_t = -K_h$
II E, t	$dh = \frac{1}{V} dE - \frac{u}{V} dt$ $dK = -\frac{u}{V} dE + \left(\frac{u^2}{V} + p\right) dt$	$V = \frac{1}{h_E}$	$u = -\frac{h_t}{h_E} = -\frac{K_E}{h_E}$ $p = \frac{1}{h_E} \frac{\partial(h, K)}{\partial(E, t)}$
III K, t	$dh = -\frac{1}{u} dK + \frac{p}{u} dt$ $dE = -\frac{V}{u} dK + \left(\frac{pV}{u} + u\right) dt$	$u = -\frac{1}{h_K}$	$p = -\frac{h_t}{h_K}$ $V = \frac{E_k}{h_K}$
IV h, E	$dt = -\frac{V}{u} dh + \frac{1}{u} dE$ $dK = -\left(u + \frac{pV}{u}\right) dh + \frac{p}{u} dE$	$u = \frac{1}{t_E}$	$V = -\frac{t_h}{t_E}$ $p = \frac{K_E}{t_E}$
V h, K	$dt = \frac{u}{p} dh + \frac{1}{p} dK$ $dE = \left(V + \frac{u^2}{p}\right) dh + \frac{u}{p} dK$	$p = \frac{1}{t_K}$	$u = \frac{t_h}{t_K} = \frac{E_k}{t_K}$ $V = \frac{1}{t_K} \frac{\partial(E, t)}{\partial(h, K)}$
VI E, K	$dh = \frac{1}{D}(pdE - udK)$ $dt = \frac{1}{D}(udE + VdK)$	$D = pV + u^2$ $u = Dt_E = -Dh_K$	$p = Dh_E$ $V = Dt_K$

It follows from (4.3) that:

$$V = \Phi_{hh}, \quad u = \Phi_{ht}, \quad p = -\Phi_{tt}, \tag{5.3}$$

where partial derivatives have been denoted again by subscripts.

The function Φ identically satisfies eq. (4.7). The remaining equation (4.8) takes the form:

$$\Phi_{tt}(\Phi_{hh})^\gamma + B(h) = 0. \tag{5.4}$$

The problem of the unsteady rectilinear motion of a perfect gas has now been reduced to the determination of a single function Φ which has to satisfy equation (5.4). In order to represent a proper flow it will in general be necessary for the function Φ to admit partial derivatives of first, second and third order, which are bounded and continuous. The second derivatives of Φ determine the flow parameters u, p and V as indicated by (5.3).

Other possibilities to reduce the problem to the determination of a single function will be considered in the following sections.

It will be possible again to substitute new independent variables (ξ, η) instead of (h, t). The appearance of the second derivatives of Φ in eq. (5.4) makes the final result complicated. Therefore we limit ourselves to applying (3.1) to the expressions for E and K in (5.2). This results into:

$$E = \frac{1}{J} \frac{\partial(\Phi, t)}{\partial(\xi, \eta)}, \qquad K = \frac{1}{J} \frac{\partial(\Phi, h)}{\partial(\xi, \eta)}. \tag{5.5}$$

Substitution of (5.5) into (4.9) then leads to the equation for $\Phi(\xi, \eta)$.

6. Legendre transformations of E and K

In section 4 the functions $E\ (= x)$ and K were introduced, which identically satisfied the equation (2.14) respectively (2.15). From the functions E and K six more functions are now easily obtained upon integration by parts. We define the functions F, G, H and L, M, N by means of:

$$\begin{aligned} F(h, u) &= -ut + E(h, t),\\ G(V, t) &= Vh - E(h, t),\\ H(V, u) &= Vh + ut - E(h, t), \end{aligned} \tag{6.1}$$

$$\begin{aligned} L(u, t) &= uh + K(h, t),\\ M(h, p) &= pt - K(h, t),\\ N(u, p) &= uh - pt + K(h, t). \end{aligned} \tag{6.2}$$

The six functions so defined are the Legendre transformations of $E(h, t)$ and $K(h, t)$. One easily checks that:

$$\begin{aligned} dF &= V dh - t du,\\ dG &= h dV - u dt,\\ dH &= h dV + t du, \end{aligned} \tag{6.3}$$

$$\begin{aligned} dL &= h du + p dt,\\ dM &= u dh + t dp,\\ dN &= h du - t dp. \end{aligned} \tag{6.4}$$

We have seen that E and K satisfy identically the first respectively the second equation of Set I in the Tables 1 and 2. The functions F, G, H, L, M, and N identically satisfy some other equations in these tables. So M identically satisfies the second equation in Set II, while F satisfies in the same way the first equation of Set III. The different functions have been denoted in the tables at the appropriate places.

Any pair of functions, one of E, F, G, H and one of K, L, M, N can be used to obtain the equations in Table 1. Consider for example Set IV with the independent variables (h, V). Starting from E and K in (4.3) it follows:

$$dE = (V + ut_h)dh + ut_v dV,$$
$$dK = (-u + pt_h)dh + pt_v dV. \tag{6.5}$$

The requirement, that the order of the differentiations in the mixed second order derivatives E_{hV} and K_{hV} can be interchanged then leads to:

$$\frac{\partial}{\partial V}(V + ut_h) = \frac{\partial}{\partial h}(ut_V),$$
$$\frac{\partial}{\partial V}(-u + pt_h) = \frac{\partial}{\partial h}(pt_V), \tag{6.6}$$

and upon further expansion to Set IV.

The function E satisfies identically the first equation of Set I in Table 1. Expressing u and p, in the second equation of Set I, in terms of E an equation for the single function E is obtained. If the equation for E is solved the flow parameters u, p and V are determined from (4.3) and (2.16) by the relations:

$$V = E_h, \quad u = E_t, \quad p = B(h)(E_h)^{-\gamma}. \tag{6.7}$$

TABLE 4

1. *Equations for the eight functions*

$E(h, t)$	$E_h^{\gamma+1} E_{tt} - \gamma B(h) E_{hh} + B'(h) E_h = 0$
$F(h, u)$	$\gamma B(h)\{F_{hh}F_{uu} - F_{hu}^2\} - B'(h) F_h F_{uu} + F_h^{\gamma+1} = 0$
$G(V, t)$	$V^{\gamma+1}\{G_{VV}G_{tt} - G_{Vt}^2\} - B'(G_V) V G_{VV} + \gamma B(G_V) = 0$
$H(V, u)$	$B'(H_V) V\{H_{VV}H_{uu} - H_{Vu}^2\} + V^{\gamma+1} H_{VV} - \gamma B(H_V) H_{uu} = 0$
$K(h, t)$	$\{B(h)\}^{1/\gamma} K_{tt} - \gamma\{K_t\}^{(\gamma+1)/\gamma} K_{hh} = 0$
$L(u, t)$	$\{B(L_u)\}^{1/\gamma}\{L_{tt}L_{uu} - L_{ut}^2\} + \gamma (L_t)^{(\gamma+1)/\gamma} = 0$
$M(h, p)$	$\gamma p^{(\gamma+1)/\gamma}\{M_{hh}M_{pp} - M_{hp}^2\} + \{B(h)\}^{1/\gamma} = 0$
$N(u, p)$	$\{B(N_u)\}^{1/\gamma} N_{uu} - \gamma p^{(\gamma+1)/\gamma} N_{pp} = 0$

TABLE 4

2. *The flow parameters p, V, u, h, t*

h, t	$u = E_t$	$V = E_h$	$p = B(h) E_h^{-\gamma}$
h, u	$V = F_h$	$t = -F_u$	$p = B(h) F_h^{-\gamma}$
V, t	$h = G_V$	$u = -G_t$	$p = B(G_V) V^{-\gamma}$
V, u	$h = H_V$	$t = H_u$	$p = B(H_V) V^{-\gamma}$
h, t	$u = -K_h$	$p = K_t$	$V^\gamma = B(h) p^{-1}$
u, t	$h = L_u$	$p = L_t$	$V^\gamma = B(L_u) L_t^{-1}$
h, p	$u = M_h$	$t = M_p$	$V^\gamma = B(h) p^{-1}$
u, p	$h = N_u$	$t = -N_p$	$V^\gamma = B(N_u) p^{-1}$

The possibility to reduce the problem to the determination of one single function exists at every place in Table 1, where one of the eight functions $E \ldots N$ appears. The eight equations obtained in this way have been listed in Table 4. Inspection shows that they are essentially of Monge–Ampère type. In the homentropic case several equations simplify. They have been listed in Table 5.

The equations for $E(h, t)$ and $K(h, t)$ can also be obtained upon eliminating K, respectively E, from the equations (4.7) and (4.8).

Since F, G and H are Legendre transformations of E, the equations for F, G and H can also be obtained by applying the appropriate Legendre transformations to the equation for E. The same applies to the equations for K, L, M and N.

The independent variables have been restricted so far, excepting section 4, to any pair of the variables u, p, V, h and t. If it were desired to extend the set of independent variables, clearly the functions E $(= x)$, F, G, H, K, L, M and N will deserve priority. In section 4 E and K already appeared as independent variables. Selecting E and t as independent variables the Eulerian equations appear. This is easily checked by employing (4.4) in the system (3.3). Taking h and x $(= E)$ as independent variables one is led in analogous fashion

TABLE 5

1. *Equations for the* 8 *functions* (*Homentropic*)

$E(h, t)$	$E_h^{\gamma+1} E_{tt} - \gamma \cdot B \cdot E_{hh} = 0$
$F(h, u)$	$\gamma \cdot B \cdot \{F_{hh} F_{uu} - F_{hu}^2\} + F_h^{\gamma+1} = 0$
$G(V, t)$	$V^{\gamma+1} \{G_{VV} G_{tt} - G_{Vt}^2\} + \gamma \cdot B = 0$
$H(V, u)$	$V^{\gamma+1} H_{VV} - \gamma \cdot B \cdot H_{uu} = 0$
$K(h, t)$	$B^{1/\gamma} K_{tt} - \gamma \{K_t\}^{(\gamma+1)/\gamma} K_{hh} = 0$
$L(u, t)$	$B^{1/\gamma} \{L_{uu} L_{tt} - L_{ut}^2\} + \gamma (L_t)^{(\gamma+1)/\gamma} = 0$
$M(h, p)$	$\gamma \cdot p^{(\gamma+1)/\gamma} \{M_{hh} M_{pp} - M_{hp}^2\} + B^{1/\gamma} = 0$
$N(u, p)$	$B^{1/\gamma} N_{uu} - \gamma \cdot p^{(\gamma+1)/\gamma} N_{pp} = 0$

TABLE 5

2. *The flow parameters* p, V, u, h, t

h, t	$u = E_t$	$V = E_h$	$p = B \cdot E_h^{-\gamma}$
h, u	$V = F_h$	$t = -F_u$	$p = B \cdot F_h^{-\gamma}$
V, t	$h = G_V$	$u = -G_t$	$p = B \cdot V^{-\gamma}$
V, u	$h = H_V$	$t = H_u$	$p = B \cdot V^{-\gamma}$
h, t	$u = -K_h$	$p = K_t$	$V^\gamma = B \cdot p^{-1}$
u, t	$h = L_u$	$p = L_t$	$V^\gamma = B \cdot L_t^{-1}$
h, p	$u = M_h$	$t = M_p$	$V^\gamma = B \cdot p^{-1}$
u, p	$h = N_u$	$t = -N_p$	$V^\gamma = B \cdot p^{-1}$

to the set of equations:

$$u\frac{\partial V}{\partial x} - V\frac{\partial u}{\partial x} - \frac{\partial u}{\partial h} = 0,$$

$$u\frac{\partial u}{\partial x} + V\frac{\partial p}{\partial x} + \frac{\partial p}{\partial h} = 0, \tag{6.8}$$

$$\frac{\partial B}{\partial x} = 0.$$

Proceeding systematically with the eight functions $E \ldots N$ sixteen new sets of equations appear. Since the function N for example is to be considered, according to (6.2) and (6.4) as a function of (u, p), two new sets are obtained, one with the independent variables (u, N), the other with the independent variables (N, p). These sixteen sets of equations have not been written out so far in a systematic way.

The transformations just discussed can also be introduced in the equations of Table 4. Two new equations will appear for every equation in the Table. From the equation for E $(= x)$ with the independent variables (h, t) one so obtains:

$$(u^2 - a^2)h_{xx} + 2uh_{xt} + h_{tt} = B'(h)V^{-(\gamma+1)}, \tag{6.9}$$

$$(u^2 - a^2)t_{xx} - \frac{2a^2}{V}t_{hx} - \frac{a^2}{V^2}t_{hh} = B'(h)u^{-1}V^{-\gamma}, \tag{6.10}$$

with $B'(h)$ denoting the derivative of $B(h)$. The sixteen equations which appear in this fashion have not been written out so far in a systematic fashion.

Some of the equations obtained here have appeared in the literature. The equation (6.9) appears in a paper of Naylor [8] and in the book of von Mises [9]. In [8] a function v of u and t is introduced, which is essentially the function $L(u, t)$ in (6.2). Naylor considers at some point u as function of L and t leading to the equation:

$$u_{LL}u_{tt} - u_{Lt}^2 + \gamma p^{(\gamma+1)/\gamma}\{B(u_L^{-1})\}^{-1/\gamma}u_L^4 = 0. \tag{6.11}$$

The derivation of this equation, one of the set of sixteen equations containing also (6.9) and (6.10) is given in an Appendix.

The equation for M in Table 4 is the equation, which formed the starting point for the work of Martin and Ludford [10]. Some further remarks on the Martin–Ludford gas appear in the section 8.

The fourth and eight equation in Table 5 are linear. They are closely related to the equation discussed by Landau and Lifshitz [11]. These authors use the velocity u and the enthalpy w as independent variables. In the homentropic case, and their discussion is restricted to this, the enthalpy is a function of p or V only and the three independent variables p, V and w are equivalent. We verify it in some detail.

The enthalpy w can be expressed in the form:

$$w = \int \frac{dp}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{\gamma}{\gamma - 1}p^{(\gamma-1)/\gamma}B^{1/\gamma} = \frac{a^2}{\gamma - 1}. \tag{6.12}$$

One easily checks

$$\frac{\partial N}{\partial p} = \frac{\partial N}{\partial w}\frac{dw}{dp} = \frac{\partial N}{\partial w} B^{1/\gamma} \cdot p^{-1/\gamma} = \frac{\partial N}{\partial w}\frac{1}{\rho},$$

$$\frac{\partial^2 N}{\partial p^2} = \frac{\partial^2 N}{\partial w^2}\left(\frac{1}{\rho}\right)^2 - \frac{\partial N}{\partial w}\frac{1}{\rho^2 a^2}.$$

The equation for $N(u, p)$ in Table 5 can be written

$$N_{uu} - \rho^2 a^2 N_{pp} = 0, \tag{6.13}$$

and replacing the derivatives to p in (6.13) by derivatives to w one finds:

$$N_{uu} - a^2 N_{ww} + N_w = 0. \tag{6.14}$$

The equation of Landau and Lifshitz, for a function of (u, w), which we denote here by χ^* is:

$$\chi^*_{uu} - (\gamma - 1) w \chi^*_{ww} - \chi^*_w = 0. \tag{6.15}$$

The equations (6.14) and (6.15) differ in the sign of the terms with first derivatives. A closer study of the equations reveals that:

$$\begin{aligned} d\chi^* &= H du - V dN, \\ dN &= \rho(H du - d\chi^*), \end{aligned} \tag{6.16}$$

with H denoting the Legendre transformation of E $(= x)$, defined in (6.1).

7. Legendre transformations of $\Phi(h, t)$

Following the method of the preceding section the Legendre Transformations of $\Phi(h, t)$ may now be considered. Writing x instead of E, we may define the functions:

$$\begin{aligned} \psi(x, t) &= hx - \Phi(h, t), \\ \theta(h, K) &= Kt + \Phi(h, t), \\ \chi(x, K) &= hx - Kt - \Phi(h, t), \end{aligned} \tag{7.1}$$

together with the differentials:

$$\begin{aligned} d\Phi &= x dh - K dt, \\ d\psi &= h dx + K dt, \\ d\theta &= x dh + t dK, \\ d\chi &= h dx - t dK. \end{aligned} \tag{7.2}$$

The function Φ identically satisfies the equation (4.7), which appears in Set I of Table 3. The three other functions satisfy identically analogous equations in the Sets II, V and VI of Table 3. The four functions have been denoted in the Table in the appropriate places.

In Section 5 an equation for Φ alone was obtained by expressing E $(= x)$ and K as derivatives of Φ and substituting these forms into (4.8). This resulted in the equation (5.4).

TABLE 6

1. *The equations for Φ, ψ, θ, χ*

h, t	$\Phi_{tt}(\Phi_{hh})^{\gamma} + B(h) = 0$
x, t	$\psi_{xt}^2 - \psi_{xx}\psi_{tt} + B(\psi_x)\psi_{xx}^{\gamma+1} = 0$
h, K	$(\theta_{hh}\theta_{KK} - \theta_{hK}^2)^{\gamma} - B(h)\theta_{KK}^{\gamma+1} = 0$
x, K	$\chi_{xx}\chi_{KK}^{\gamma} + B(\chi_x)(\chi_{xx}\chi_{KK} - \chi_{xK}^2)^{\gamma+1} = 0$

TABLE 6

2. *The flow parameters*

h, t	$x = \Phi_h$ $K = -\Phi_t$	$V = \Phi_{hh}$ $u = \Phi_{ht}$	$p = -\Phi_{tt}$
x, t	$h = \psi_x$ $K = \psi_t$	$V = \dfrac{1}{\psi_{xx}}$ $u = -\dfrac{\psi_{xt}}{\psi_{xx}}$	$p = \dfrac{\psi_{tt}\psi_{xx} - \psi_{xt}^2}{\psi_{xx}}$
h, K	$x = \theta_h$ $t = \theta_K$	$V = \dfrac{\theta_{hh}\theta_{KK} - \theta_{hK}^2}{\theta_{KK}}$ $u = \dfrac{\theta_{hK}}{\theta_{KK}}$	$p = \dfrac{1}{\theta_{KK}}$
x, K	$h = \chi_x$ $t = -\chi_K$	$V = \dfrac{1}{D}\chi_{KK}$ $u = \dfrac{1}{D}\chi_{xK}$	$p = -\dfrac{1}{D}\chi_{xx}$ $D = \chi_{xx}\chi_{KK} - \chi_{xK}^2$

For the sets II, V and VI of Table 3 the same possibilities exist, leading to formulation of the problem in terms of the single functions ψ, θ or χ.

The four equations which appear in that way have been listed in the first part of Table 6. In the second part of Table 5 the flow parameters have been expressed in the relevant derivatives. For $\Phi(h, t)$ these parameters already appeared in (5.3).

Since the functions ψ, θ and χ are Legendre transformations of Φ, the equations for ψ, θ and χ can also be obtained by application of the appropriate Legendre transformation to eq. (5.4). The equation for ψ has appeared in the literature and was introduced by Smith [12].

Some other expressions appear if we express dx and dK in (7.2) by means of (4.3). Instead of (7.2) one obtains:

$$\begin{aligned} d\Phi &= x\,dh - K\,dt, \\ d\psi &= hV\,dh + (hu + K)\,dt, \\ d\theta &= (x - ut)\,dh + pt\,dt, \\ d\chi &= (hV + ut)\,dh + (uh - tp)\,dt. \end{aligned} \tag{7.3}$$

Writing out the mixed second derivatives with respect to h and t of the four functions $\Phi - \chi$ then leads to the four equations in conservation form:

$$\begin{aligned}
&\frac{\partial x}{\partial t} + \frac{\partial K}{\partial h} = 0,\\
&\frac{\partial}{\partial t}(hV) - \frac{\partial}{\partial h}(uh + K) = 0,\\
&\frac{\partial}{\partial t}(x - ut) - \frac{\partial}{\partial h}(pt) = 0,\\
&\frac{\partial}{\partial t}(hV + ut) - \frac{\partial}{\partial h}(uh - pt) = 0.
\end{aligned} \tag{7.4}$$

Expanding the relations in (7.4) they appear as combinations of the equations (2.14) and (2.15) for the conservation of mass and momentum, together with the definitions (4.3) of x and K.

The steps leading to (7.3) and (7.4) can easily be repeated, when the independent variables (h, t) are replaced by the independent variables (x, t), (h, K) and (x, K). Analogous expressions can then be obtained. Comparison of (7.3) with the definition in (6.1) and (6.2) shows that the increments in (7.3) can also be written:

$$\begin{aligned}
d\Phi &= E\,dh - K\,dt,\\
d\psi &= (G + E)dh + L\,dt,\\
d\theta &= F\,dh + (M + K)dt,\\
d\chi &= (H + E)dh + (N + K)dt.
\end{aligned} \tag{7.5}$$

The functions E, F, G, H, K, L, M and N may then be written as derivatives with respect to h and t of the functions $\Phi \ldots \chi$.

One finds:

$$\begin{aligned}
&E = \Phi_h, \quad F = \theta_h, \quad G = \psi_h - \Phi_h, \quad H = \chi_h - \Phi_h,\\
&K = -\Phi_t, \quad L = \psi_t, \quad M = \Phi_t + \theta_t, \quad N = \Phi_t + \chi_t.
\end{aligned} \tag{7.6}$$

Further possibilities arise if also the expressions (6.3) and (6.4) are admitted for substitution into (7.2). All these steps have not been pursued in detail so far.

Finally we mention the possibilities to interchange dependent and independent variables. With the equation for $\chi(x, K)$ two new equations can then be constructed, one for $x(\chi, K)$, the other for $K(x, \chi)$. Altogether 8 new equations will then be obtained.

8. The characteristics of equations in Table 1

In this section we wish to consider the characteristics of the ten Sets of equations in Table 1. In section 2 the characteristic equations for Set I were found to be (2.20) and (2.21). For the other nine Sets in the Table the characteristic equations are exactly the same. It will be demonstrated for Set IV with the independent variables h, V. For the other systems the same methods apply.

Set IV represents a system of two simultaneous equations for the determination of u and t. The parameter p is known in terms of h and V from (2.16), while logarithmic differentiation of (2.16) and employing (2.18) leads to:

$$dp + \frac{a^2}{V^2} dV = \frac{pB'(h)\,dh}{B(h)}. \tag{8.1}$$

The characteristic equations will be determined from the initial value problem. Let ξ, η denote a set of curvilinear coordinates in the h, V-plane. The coordinates h and V are known in terms of ξ and η, while also the converse applies. Let initial data of u and t be given along a segment of the curve $\xi = 0$. Along this segment u and t are functions of η only. The question is to what extent the partial differential equations together with the initial data will determine the outward derivatives $\partial u/\partial\xi$ and $\partial t/\partial\xi$, which are required for the determination of u and t outside the initial curve.

The Set IV in the curvilinear coordinates ξ, η is given by (3.5). In view of the derivatives to be considered they can be written in the form:

$$\begin{aligned} t_\eta \frac{\partial u}{\partial \xi} - u_\eta \frac{\partial t}{\partial \xi} &= h_\xi V_\eta - V_\xi h_\eta, \\ h_\eta \frac{\partial u}{\partial \xi} + p_\eta \frac{\partial t}{\partial \xi} &= h_\xi u_\eta + t_\eta p_\xi. \end{aligned} \tag{8.2}$$

The derivatives of h, V with respect to ξ and η are known from the definition of the curvilinear coordinates. By means of (8.1) these derivatives determine the derivatives of p. Finally u_η and t_η are known from the initial data along the initial segment of the curve $\xi = 0$.

It is clear from (8.2) that the partial derivatives $\partial u/\partial\xi$ and $\partial t/\partial\xi$ are uniquely determined along the initial segment of $\xi = 0$ when the determinant D, with:

$$D = \begin{vmatrix} t_\eta & -u_\eta \\ h_\eta & p_\eta \end{vmatrix} = t_\eta p_\eta + u_\eta h_\eta, \tag{8.3}$$

is different from zero in each point of the initial segment. In a point with $D = 0$ there will be no solution for $\partial u/\partial\xi$ and $\partial t/\partial\xi$. An infinity of solutions however will exist in that case if the matrix of the system (8.2) has rank 1. In that case we have:

$$\frac{t_\eta}{h_\eta} = \frac{-u_\eta}{p_\eta} = \frac{h_\xi V_\eta - V_\xi h_\eta}{h_\xi u_\eta + t_\eta p_\xi}. \tag{8.4}$$

If the ratios in (8.4) are put equal to λ and V is eliminated by using (8.1), the set (8.4) can be written in the form:

$$\begin{gathered} t_\eta - \lambda h_\eta = 0, \qquad u_\eta + \lambda p_\eta = 0, \\ h_\xi\left(\lambda u_\eta + \frac{V^2}{a^2} p_\eta\right) + p_\xi\left(\lambda t_\eta - \frac{V^2}{a^2} h_\eta\right) = 0. \end{gathered} \tag{8.5}$$

The sets (8.4) and (8.5) determine the characteristic directions and the characteristic equations. It is not difficult to see then, that the three equations can be satisfied simultaneously only provided $\lambda^2 = V^2/a^2$ and $\lambda = \pm V/a$. Substitution of these two values of λ leads to the two sets of characteristic equations:

$$\left.\begin{aligned} t_\eta - \frac{V}{a} h_\eta &= 0, \\ u_\eta + \frac{V}{a} p_\eta &= 0, \end{aligned}\right\} \tag{8.6}$$

$$\left.\begin{aligned} t_\eta + \frac{V}{a} h_\eta &= 0, \\ u_\eta - \frac{V}{a} p_\eta &= 0, \end{aligned}\right\} \tag{8.7}$$

which are identical with (2.20) and (2.21).

It may be verified that the characteristic equations for the other sets in Table 1 are identical with (2.20), (2.21) and (8.6), (8.7).

The characteristic equations determine the slopes of the characteristics if one works in the h, t-plane (Set I) or the u, p-plane (Set VIII). In the other cases the slopes of the characteristics are not determined so simply.

In the non-homentropic case the characteristic acoustic impedance a/V depends on p and h or V and h as shown in (2.18). In the homentropic case a/V can be considered as a function of p alone, (or V alone). In that case we have:

$$du \pm \frac{V}{a} dp = du \pm \frac{a}{\gamma p} dp = d\left(u \pm \frac{2a}{\gamma - 1}\right) = 0, \tag{8.8}$$

leading to the well-known Riemann invariants:

$$r = u + \frac{2a}{\gamma - 1}, \quad s = u - \frac{2a}{\gamma - 1}, \tag{8.9}$$

which are constant along the r-characteristics, with slope $dh/dt = a/V$ in the h, t-plane, respectively the s-characteristics with slope $dh/dt = -a/V$ in the h, t-plane. For the discussion of the homentropic flows the Riemann invariants are of crucial importance.

We consider next the Set VI in Table 1 with the independent variables u, t. Associated with this set of non-homentropic equations is the function L, defined in (6.2), with the differential:

$$dL = h\,du + p\,dt. \tag{8.10}$$

Considering this relation in the (characteristic) direction in the u, t-plane, associated with the characteristics (2.20) or (8.6), it may be rewritten in the form:

$$\begin{aligned} 0 &= dL - h\,du - p\,dt = dL + \frac{V}{a}(h\,dp - p\,dh) \\ &= dL + h^2 \frac{V}{a} d\left(\frac{p}{h}\right) = dL - p^2 \frac{V}{a} d\left(\frac{h}{p}\right). \end{aligned} \tag{8.11}$$

Substitution of (2.18) then gives:

$$0 = dL - \frac{1}{\sqrt{\gamma}}\{B(h)\}^{1/2\gamma} p^{(3\gamma-1)/2\gamma} d\left(\frac{h}{p}\right). \tag{8.12}$$

Selecting $B(h)$ to be of the form:

$$B(h) = C^{2\gamma} h^{-(3\gamma-1)}, \tag{8.13}$$

with C a constant, the second term in (8.12) depends on h/p only and (8.12) can be rewritten:

$$d\left\{L - \frac{2C\sqrt{\gamma}}{\gamma - 1}\left(\frac{p}{h}\right)^{(\gamma-1)/2\gamma}\right\} = 0, \tag{8.14}$$

indicating that the expression in braces is constant along the characteristics determined by (2.20) or (8.6). In analogy with the homentropic case the expression in (8.14) is called the generalized Riemann invariant r. By the same steps, applied to the characteristic equations (2.21) or (8.7), the second generalized Riemann invariant s may be obtained and we have finally:

$$\begin{aligned} r &= L - \frac{2C\sqrt{\gamma}}{\gamma - 1}\left(\frac{p}{h}\right)^{(\gamma-1)/2\gamma} = K - h\left(u + \frac{2a}{\gamma - 1}\right), \\ s &= L + \frac{2C\sqrt{\gamma}}{\gamma - 1}\left(\frac{p}{h}\right)^{(\gamma-1)/2\gamma} = K - h\left(u - \frac{2a}{\gamma - 1}\right), \end{aligned} \tag{8.15}$$

which are constant along the r-characteristics, respectively s-characteristics in a gas with the entropy distribution determined from (8.13). This gas will be called a Martin–Ludford gas after the authors who first explored this case [10].

Since the characteristic equations for all the sets in Table 1 are identical it is likely that the Martin–Ludford gas is the only case, where generalized Riemann invariants of the type considered here, will exist in non-homentropic flows. Different ways of generalizing Riemann invariants have been considered by P. D. Lax [13, 14].

In the two cases we considered, the homentropic case and the Martin–Ludford gas, the Riemann invariants may be chosen as independent variables. For the Martin–Ludford gas one may consult [10]. For the homentropic case the references [1, 2, 3] may be considered. In the latter case one finds the equations:

$$\begin{aligned} (r - s)\frac{\partial^2 t}{\partial r \partial s} - \frac{\gamma + 1}{2(\gamma - 1)}\left(\frac{\partial t}{\partial r} - \frac{\partial t}{\partial s}\right) &= 0, \\ (r - s)\frac{\partial^2 h}{\partial r \partial s} + \frac{\gamma + 1}{2(\gamma - 1)}\left(\frac{\partial h}{\partial r} - \frac{\partial h}{\partial s}\right) &= 0. \end{aligned} \tag{8.16}$$

These are linear hyperbolic equations of the Euler–Poisson–Darboux-type. In particular the equation for $t(r, s)$ has been used repeatedly. We like to show that the equation for $t(r, s)$ is closely related with the linear equations in Table 5 for $N(u, p)$ and $H(V, u)$ and the equation (6.15) of Landau and Lifshitz.

From the definition of dN in (6.4) it follows that $t = -N_p$. Differentation of the equation for N in Table 5, with respect to p then gives:

$$B^{1/\gamma}t_{uu} - \gamma p^{(\gamma+1)/\gamma}t_{pp} - (\gamma+1)p^{1/\gamma}t_p = 0. \tag{8.17}$$

Replacing the variable p by the speed of sound we find:

$$\frac{\partial t}{\partial p} = \frac{\gamma - 1}{2}\,\frac{V}{a}\,\frac{\partial t}{\partial a},$$

$$\frac{\partial^2 t}{\partial p^2} = \left(\frac{\gamma-1}{2}\right)^2 \frac{V^2}{a^2}\,\frac{\partial^2 t}{\partial a^2} - \frac{\gamma^2-1}{4}\,\frac{V^2}{a^3}\,\frac{\partial t}{\partial a}.$$

Putting:

$$\bar{a} = \frac{2a}{\gamma - 1} = \tfrac{1}{2}(r - s),$$

the equation (8.17) now takes the form:

$$\frac{\partial^2 t}{\partial u^2} - \frac{\partial^2 t}{\partial \bar{a}^2} - \frac{\gamma+1}{\gamma-1}\,\frac{1}{\bar{a}}\,\frac{\partial t}{\partial \bar{a}} = 0. \tag{8.18}$$

Since we have from (8.9)

$$r = u + \bar{a}, \qquad s = u - \bar{a},$$

it is easy to replace u, $\bar{a}$ by the variables r, s and to retrieve the equation for t in (8.16).

In similar fashion the equation for $H(V, u)$ and (6.13) can be dealt with.

9. Final remarks

In this paper a systematic study has been made of the equations of motion for the unsteady rectilinear motion of a perfect gas. By allowing some freedom in the choice of independent variables, and by employing potentials or streamfunctions, a considerable number of different forms of the equations are obtained. Some of these have been employed before. The present systematic analysis gives some insight in the mathematical structure of the problem.

In particular it is found that the equations take very simple forms in the Lagrangian variables (h, t). This applies whether one works with the dependent variables p, V, u (Section 2), with the "first order potentials" (Sections 4 and 6) or with the "second order potentials" (Sections 5 and 7).

A preliminary version of this paper together with some other material, but without the "second order potentials" Φ, ψ, θ, χ has appeared in Report VTH-168, June 1971 [15]. Some errors in this Report have been corrected (eq. (6.10) and the discussion in connection with eqs. (6.12)–(6.16)).

Appendix 1. Derivation of the equation (6.11) for $u(L, t)$

From (6.4) we have:

$$dL = h\,du + p\,dt, \quad L_u = h, \quad L_t = p. \tag{A.1}$$

Rearranging this equation gives:

$$du = \frac{1}{h}\,dL - \frac{p}{h}\,dt, \quad u_L = \frac{1}{h}, \quad u_t = -\frac{p}{h}. \tag{A.2}$$

From (A.1) and (A.2) we deduce:

$$L_u = \frac{1}{u_L}, \quad L_t = p = -hu_t = -\frac{u_t}{u_L}. \tag{A.3}$$

From the first relation in (A.3) it follows:

$$dL_u = L_{uu}du + L_{ut}dt = -\frac{1}{u_L^2}\,du_L = -\frac{1}{u_L^2}(u_{LL}dL + u_{Lt}dt). \tag{A.4}$$

Substituting dL from (A.1) in (A.4) then leads to:

$$L_{uu} = -\frac{u_{LL}}{u_L^3}, \quad L_{ut} = \frac{u_t u_{LL} - u_L u_{Lt}}{u_L^3}. \tag{A.5}$$

In the same way the second relation in (A.3) leads to:

$$\begin{aligned} dL_t = L_{tu}du + L_{tt}dt &= -\frac{1}{u_L}(u_{tL}dL + u_{tt}dt) + \frac{u_t}{u_L^2}(u_{LL}dL + u_{Lt}dt) \\ &= \frac{u_t u_{LL} - u_L u_{Lt}}{u_L^2}\,dL + \frac{u_t u_{tL} - u_L u_{tt}}{u_L^2}\,dt. \end{aligned} \tag{A.6}$$

Substituting dL from (A.1) in (A.6) then gives:

$$\begin{aligned} L_{ut} &= \frac{u_t u_{LL} - u_L u_{Lt}}{u_L^3}, \\ L_{tt} &= -\frac{u_t^2 u_{LL} - 2u_L u_t u_{Lt} + u_L^2 u_{tt}}{u_L^3}. \end{aligned} \tag{A.7}$$

One easily checks that:

$$L_{tt}L_{uu} - L_{ut}^2 = \frac{u_{tt}u_{LL} - u_{Lt}^2}{u_L^4}. \tag{A.8}$$

Substitution of (A.3) and (A.8) into the equation for L in Table 4 then leads to:

$$\left\{B\left(\frac{1}{u_L}\right)\right\}^{1/\gamma} \frac{u_{tt}u_{LL} - u_{Lt}^2}{u_L^4} + \gamma\left(-\frac{u_t}{u_L}\right)^{(\gamma+1)/\gamma} = 0, \tag{A.9}$$

which can be easily reduced to the form (6.11).

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